J. reine angew. Math. **648** (2010), 69—73
DOI 10.1515/CRELLE.2010.079

Journal für die reine und angewandte Mathematik

Knotted holomorphic discs in $\mathbb{C}^2$

By *S. Baader* at Zürich, *F. Kutzschebauch* at Bern, and *E. F. Wold* at Oslo

---

Abstract. We construct knotted proper holomorphic embeddings of the unit disc in $\mathbb{C}^2$.

1. Introduction

Every classical knot type can be represented by a polynomial embedding of $\mathbb{R}$ in $\mathbb{R}^3$ [9]. In particular, there exist topologically distinct polynomial embeddings of $\mathbb{R}$ in $\mathbb{R}^3$. Crossing these with another coordinate, we obtain topologically distinct polynomial embeddings of $\mathbb{R}^2$ in $\mathbb{R}^4$. In contrast, all polynomial embeddings of $\mathbb{C}$ in $\mathbb{C}^2$ are topologically equivalent, in fact even connected by a polynomial automorphism of $\mathbb{C}^2$. The first complete algebraic proof of this fact is due to Abhyankar and Moh [1]; later a purely knot theoretical proof was found by Rudolph [6]. It is an open question whether proper holomorphic embeddings of the complex plane $\mathbb{C}$ or of the unit disk $\triangle \subset \mathbb{C}$ in $\mathbb{C}^2$ can be topologically knotted (see Problem 1.102 (A/B) of Kirby's list [4]). In this note we construct knotted proper holomorphic embeddings of the unit disc in $\mathbb{C}^2$.

Theorem 1. *There exist topologically knotted proper holomorphic embeddings of the unit disc in* $\mathbb{C}^2$.

Since the image of a holomorphically embedded disc in $\mathbb{C}^2$ is a minimal surface, we obtain the following corollary, which solves Problem 1.102 (C) of Kirby's list.

Corollary 1. *There exists a proper embedding* $f : \mathbb{R}^2 \to \mathbb{R}^4$ *whose image is a topologically knotted complete minimal surface.*

Our construction is based on the existence of locally well-behaved Fatou-Bieberbach domains and on the existence of knotted holomorphic discs in the 4-ball. A Fatou-Bieberbach domain in $\mathbb{C}^2$ is an open subset of $\mathbb{C}^2$ which is biholomorphically equivalent to $\mathbb{C}^2$. Fatou-Bieberbach domains tend to have wild shapes; the first Fatou-Bieberbach domain with smooth boundary was constructed by Stensønes [10]. In [3], Globevnik constructed Fatou-Bieberbach domains $\Omega \subset \mathbb{C}^2$ whose intersections with $\mathbb{C} \times \Delta$ are arbitrarily small $\mathscr{C}^1$-perturbations of $\Delta \times \Delta$. Here Δ stands for the open unit disc in $\mathbb{C}$. These domains are well-adapted for constructing knotted holomorphic discs in $\mathbb{C}^2$. Indeed, all we need is a knotted proper holomorphic embedding of the closed unit disc $\varphi : \overline{\Delta} \to \overline{\Delta} \times \overline{\Delta}$ that maps $\partial\overline{\Delta}$

to $\partial\overline{\Delta} \times \Delta$. Composing the restriction of this embedding $\varphi|_\Delta : \Delta \to \Delta \times \Delta \subset \Omega$ with a biholomorphism $h : \Omega \to \mathbb{C}^2$ yields a knotted proper holomorphic embedding of Δ in $\mathbb{C}^2$. The existence of knotted proper holomorphic embeddings $\varphi : \overline{\Delta} \to \overline{\Delta} \times \overline{\Delta}$ is easily established by using the theory of complex algebraic curves in $\mathbb{C}^2$.

We describe the relevant features of the Fatou-Bieberbach domains needed for our purpose in Section 2. The proof of Theorem 1 is completed in Section 3, where we give an explicit example of a knotted holomorphic disc in $\overline{\Delta} \times \overline{\Delta}$.

2. Fatou-Bieberbach domains

Fatou-Bieberbach domains in $\mathbb{C}^2$ are usually described by certain infinite processes, for example as domains of convergence of maps defined by sequences of automorphisms. This technique was used by Globevnik to construct Fatou-Bieberbach domains with controlled shape inside $\mathbb{C} \times \Delta$.

Theorem 2 (Globevnik [3]). *Let $Q \subset \mathbb{C}$ be a bounded open set with boundary of class $\mathscr{C}^1$ whose complement is connected. Let $0 < R < \infty$ be such that $\overline{Q} \subset R\Delta$. There are a domain $\Omega \subset \mathbb{C}^2$ and a volume-preserving biholomorphic map from Ω onto $\mathbb{C}^2$ such that*:

(i) $\Omega \subset \{(z, w) \in \mathbb{C}^2 : |z| < \max\{R, |w|\}\}$.

(ii) $\Omega \cap R(\Delta \times \Delta)$ *is an arbitrarily small $\mathscr{C}^1$-perturbation of $Q \times R\Delta$.*

The assumptions of Theorem 2 are verified when Q is an open disc with smooth boundary (and $R > 0$ large enough). This is the version we will need. As mentioned in the introduction, we will insert knotted discs in $\Delta \times \Delta$ with boundary in $\partial\overline{\Delta} \times \Delta$. It is a priori not clear whether such discs stay knotted in the larger domain Ω. The following lemma allows us to control knottedness on the level of the fundamental group.

Lemma 1. *Let $\Omega \subset \mathbb{C}^2$ be an open domain homeomorphic to $\mathbb{C}^2$ with*

$$\Omega \cap \mathbb{C} \times \Delta = \Delta \times \Delta,$$

and let $X \subset \Delta \times \Delta$ be a subset with $\overline{X} \subset \overline{\Delta} \times \Delta$. The inclusion $i : (\Delta \times \Delta)\backslash X \to \Omega\backslash X$ induces an injective map $i_* : \pi_1((\Delta \times \Delta)\backslash X) \to \pi_1(\Omega\backslash X)$ (suppressing base points).

Remark. At this point it does not matter whether $\Omega \cap \mathbb{C} \times \Delta$ is precisely $\Delta \times \Delta$ or a small $\mathscr{C}^1$-perturbation of it. Further, the corresponding statement stays true if the intersection $\Omega \cap (\mathbb{C} \times \Delta)$ is a small $\mathscr{C}^1$-perturbation of $D \times \Delta$, where $D \subset \mathbb{C}$ is any embedded disc with smooth boundary.

Proof. Choose $\varepsilon > 0$ such that $X \subset \Delta \times (1 - \varepsilon)\Delta$. Setting

$$U := (\Delta \times \Delta)\backslash X$$

and

$$V := \Omega\backslash(\Delta \times \overline{(1 - \varepsilon)\Delta}),$$

we have $\Omega\backslash X = U \cup V$. The theorem of Seifert-van Kampen tells us that $\pi_1(\Omega\backslash X)$ is the free product of $\pi_1(U)$ and $\pi_1(V)$ amalgamated over $\pi_1(U \cap V)$. The latter is isomorphic to $\mathbb{Z}$, since

$$U \cap V = \Delta \times \big(\Delta\backslash\overline{(1-\varepsilon)\Delta}\big)$$

is homotopy equivalent to a circle. Let $\gamma : S^1 \to U \cap V$ be a loop that generates $\pi_1(U \cap V)$. If the induced map $i_* : \pi_1\big((\Delta \times \Delta)\backslash X\big) \to \pi_1(\Omega\backslash X)$ were not injective, then a certain non-zero multiple of $[\gamma] \in \pi_1(V)$ would have to vanish. This is impossible since the inclusion $j : V \to \mathbb{C} \times \mathbb{C}^*$ maps γ onto a generator of $\pi_1(\mathbb{C} \times \mathbb{C}^*) \cong \mathbb{Z}$. $\square$

3. Complex plane curves

A complex plane curve is the zero level $V_f = \{(z,w) \in \mathbb{C}^2 : f(z,w) = 0\} \subset \mathbb{C}^2$ of a non-constant polynomial $f(z,w) \in \mathbb{C}[z,w]$. Complex plane curves form a rich source of examples in algebraic geometry and topology. For example, the intersection of a complex plane curve V_f with the boundary of a small ball centred at an isolated singularity of f forms a link which is often called algebraic. The class of algebraic links has been generalized by Rudolph to the larger class of quasipositive links [7]. Using the theory of quasipositive links, it is easy to construct knotted proper holomorphic embedded discs in $\Delta \times \Delta$. In the following, we will give a short description of Rudolph's theory; more details are contained in [7] and [8].

Let $f(z,w) = f_0(z)w^n + f_1(z)w^{n-1} + \cdots + f_n(z) \in \mathbb{C}[z,w]$ be a non-constant polynomial, $f_0(z) \neq 0$. Under some generic conditions on f, the set B of complex numbers z such that the equation $f(z,w) = 0$ has strictly less than n solutions w, is finite. Let $\gamma \subset \mathbb{C}\backslash B$ be a smooth simple closed curve. The intersection $L = V_f \cap \gamma \times \mathbb{C}$ is a smooth closed 1-dimensional manifold, i.e. a link, in the solid torus $\gamma \times \mathbb{C}$. More precisely, the link L is an n-stranded braid in $\gamma \times \mathbb{C}$, which becomes a link in S^3 via a standard embedding of $\gamma \times \mathbb{C}$ in S^3. Links that arise in this way are called quasipositive.

If $D \subset \mathbb{C}$ is the closed disc bounded by γ, then the intersection $X = V_f \cap D \times \mathbb{C}$ is a piece of complex plane curve bounded by the link L. Choosing the polynomial f and the curve γ appropriately, it is possible to arrange X to be an embedded disc with a non-trivial quasipositive knot as boundary (see [7], Example 3.2). Assuming that X is compact, there exists $R > 0$, such that $X \subset D \times R\Delta$. In order to establish Theorem 1, it remains to find an example where X is knotted in $D \times R\Delta$, more precisely

$$\pi_1(D \times R\Delta\backslash X) \ncong \mathbb{Z}.$$

In view of Lemma 1, this will then give rise to a knotted proper holomorphic embedding of the unit disc in Ω, hence in $\mathbb{C}^2$.

Remark. Here we choose $\Omega \subset \mathbb{C}^2$ to be a Fatou-Bieberbach domain whose intersection $\Omega \cap \mathbb{C} \times R\Delta$ is a small $\mathscr{C}^1$-perturbation of $D \times R\Delta$. If this perturbation is small enough, then the intersection $\tilde{X} = V_f \cap \mathbb{C} \times R\Delta \cap \Omega$ is still a proper holomorphic embedded disc, as knotted as X in $D \times R\Delta$.

We conclude this section with a concrete example, in fact Rudolph's Example 3.2 of [7]. Let

$$f(z,w) = w^3 - 3w + 2z^4.$$

The complex plane curve V_f is easily seen to be non-singular. The equation $f(z_0, w) = 0$ fails to have three distinct solutions w, if and only if the two equations $f(z_0, w) = 0$ and $\frac{\partial}{\partial w} f(z_0, w) = 3w^2 - 3 = 0$ have a simultaneous solution. This happens precisely when $z_0^8 = 1$. Thus the set B consists of the 8th roots of unity. For $z \in \mathbb{C} \backslash B$, the equation $f(z, w) = 0$ has three distinct solutions $w_1, w_2, w_3 \in \mathbb{C}$, which we index by increasing real parts. Further, it will be convenient to determine the set B_+ of complex numbers z such that the equation $f(z, w) = 0$ has two distinct solutions w with coinciding real parts. In our example, B_+ consists of 8 rays emanating from the points of B, as shown in Figure 1. These rays carry labels 1 or 2, depending on whether the real parts of w_1 and w_2 or those of w_2 and w_3 coincide. There is a way of orienting the rays that corresponds to choosing positive standard generators of the braid group. Figure 1 also indicates a curve γ that bounds a disc D. We claim that the piece of complex plane curve $X = V_f \cap D \times \mathbb{C}$ is a disc with $\pi_1\big((D \times \mathbb{C}) \backslash X\big) \ncong \mathbb{Z}$.

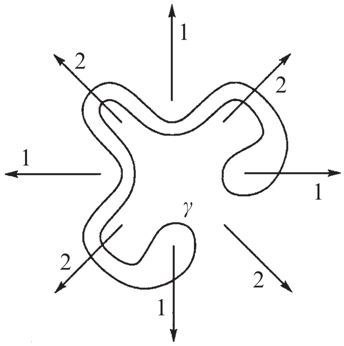

Figure 1

First, we observe that if we 'cut off' the two ends of the disc D of Figure 1, we obtain another disc D' disjoint from B. The intersection $V_f \cap D' \times \mathbb{C}$ is therefore a disjoint union of three discs. Adding the ends to D' again gives rise to two identifications along the boundaries of these discs. It is easy to see that these identifications result in a single disc whose boundary $L = V_f \cap (\gamma \times \mathbb{C})$ is knotted (actually L is the quasipositive ribbon knot 8_{20}). However, the mere fact that L is knotted does not imply that the fundamental group of $(D \times \mathbb{C}) \backslash X$ is not isomorphic to $\mathbb{Z}$ (see the last paragraph of [2]). We will describe $\pi_1\big((D \times \mathbb{C}) \backslash X\big)$ explicitly, in terms of generators and relations. Hereby we will use Orevkov's method for presenting the fundamental group of the complement of complex plane curves in $\mathbb{C}^2$ [5]. To every connected component of $\mathbb{C} \backslash B_+$ (in our case $D \backslash B_+$), we assign n (in our case 3) generators corresponding to meridians of the n discs lying over that component. Every oriented edge of the graph B_+ gives rise to n relations between the generators of the two (not necessarily distinct) components adjacent to that edge. All these relations are of Wirtinger-type (see [5], Lemma 3.1).

Let us denote the four connected components of $D \backslash B_+$ by U_1, U_2, U_3, U_4, starting at the bottom left and going clockwise around D. To every region U_j correspond three generators $\alpha_{ij} = \alpha_i(U_j)$, $1 \leqq i \leqq 3$. The relations among these generators read as follows:

1. edge: $\alpha_{11} = \alpha_{21}$.

2. edge: $\alpha_{11} = \alpha_{12}$, $\alpha_{31} = \alpha_{22}$, $\alpha_{21} = \alpha_{31}\alpha_{32}\alpha_{31}^{-1}$.

3. edge: $\alpha_{12} = \alpha_{13}$, $\alpha_{32} = \alpha_{23}$, $\alpha_{22} = \alpha_{32}\alpha_{33}\alpha_{32}^{-1}$.

4. edge: $\alpha_{13} = \alpha_{14}$, $\alpha_{33} = \alpha_{24}$, $\alpha_{23} = \alpha_{33}\alpha_{34}\alpha_{33}^{-1}$.

5. edge: $\alpha_{14} = \alpha_{24}$.

From this it is easy to verify that the assignment $\alpha_{11} \mapsto (23)$, $\alpha_{31} \mapsto (12)$ defines a surjective homomorphism $\varphi : \pi_1\big((D \times \mathbb{C}) \backslash X\big) \to S_3$, whence $\pi_1\big((D \times \mathbb{C}) \backslash X\big)$ is not a cyclic group.

References

[1] *S. S. Abhyankar* and *T. T. Moh*, Embeddings of the line in the plane, J. reine angew. Math. **276** (1975), 148–166.

[2] *S. Friedl* and *P. Teichner*, New topologically slice knots, Geom. Topol. **9** (2005), 2129–2158.

[3] *J. Globevnik*, On Fatou-Bieberbach domains, Math. Z. **229** (1998), no. 1, 91–106.

[4] *R. Kirby*, Problems in Low-Dimensional Topology, www.math.berkeley.edu/~kirby/.

[5] *S. Yu. Orevkov*, The fundamental group of the complement of a plane algebraic curve (Russian), Mat. Sbornik **137** (**179**) (1988), no. 2, 260–270; English version: http://picard.ups-tlse.fr/~orevkov/.

[6] *L. Rudolph*, Embeddings of the line in the plane, J. reine angew. Math. **337** (1982), 113–118.

[7] *L. Rudolph*, Algebraic functions and closed braids, Topology **22** (1983), no. 2, 191–202.

[8] *L. Rudolph*, Knot theory of complex plane curves, Handbook of knot theory, Elsevier B. V., Amsterdam (2005), 349–427.

[9] *A. R. Shastri*, Polynomial representations of knots, Tohoku Math. J. (2) **44** (1992), no. 1, 11–17.

[10] *B. Stensønes*, Fatou-Bieberbach domains with C^∞-smooth boundary, Ann. Math. (2) **145** (1997), no. 2, 365–377.

ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland
e-mail: sebastian.baader@math.ethz.ch

Universität Bern, Sidlerstrasse 5, 3012 Bern, Switzerland
e-mail: Frank.Kutzschebauch@math.unibe.ch

Universitetet I Oslo, Postboks 1053 Blindern, 0316 Oslo, Norway
e-mail: erlendfw@math.uio.no

Eingegangen 29. August 2008